

# A new geometrical look at Ostrogradsky procedure.

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Making use of the modern techniques of non-holonomic geometry and constrained variational calculus, a revisitation of Ostrogradsky's Hamiltonian formulation of the evolution equations determined by a Lagrangian of order  $\geq 2$  in the derivatives of the configuration variables is presented.

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## I. INTRODUCTION

About twenty years after the first formulation of Hamiltonian mechanics, Ostrogradsky proposed a generalisation of Hamilton's procedure, valid for Lagrangians involving derivatives of any order of the configuration variables [1, 3, 7].

In recent years, the study of this type of Lagrangians has been reconsidered in the context of gravitational physics and, in particular, in the development of a theoretical framework for inflation and dark energy based on modifications of General Relativity (see e.g. [2, 4, 5]).

Despite this renewed interest, to the best of the authors' knowledge, a precise geometric interpretation of Ostrogradsky's construction is still missing. In an attempt to fill this gap, we propose here a reformulation of Ostrogradsky's formalism in modern geometrical terms.

Given the event space, meant as a fibre bundle  $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ , we regard the  $N^{\text{th}}$  jet bundle  $j_N(\mathcal{V}_{n+1})$  as an affine subbundle of the first jet  $j_1(j_{N-1}(\mathcal{V}_{n+1}))$  [13, 14]. In this way, any problem involving a Lagrangian depending on the derivatives of order  $\leq N$  of the configuration variables is converted into an ordinary *constrained* variational problem.

The problem is then analysed, making use of a revisitation of Pontryagin's *maximum principle* recently developed in [8] (in this connection, see also [11, 12] and references therein). In the case of a non-degenerate Lagrangian  $L(t, q^k, \dot{q}^k, \ddot{q}^k, \dots)$ , the algorithm picks out a natural concept of "phase space", identifying it with a submanifold  $\mathcal{S}$  of the contact bundle over  $j_N(\mathcal{V}_{n+1})$ , uniquely determined by the Pontryagin Hamiltonian associated with  $L$ .

In the resulting environment, the canonical momenta and the Ostrogradsky Hamiltonian are simply the pull-back of the coordinate functions along the fibres of the contact bundle and of the Pontryagin Hamiltonian, while the Ostrogradsky equations reproduce the Hamilton–Pontryagin equations associated with the constrained variational problem.

The layout of the paper is the following: in Section II, the geometrical setup for constrained variational calculus is briefly reviewed; Section III is then devoted to the geometric reformulation of the Ostrogradsky procedure.

## II. CONSTRAINED VARIATIONAL CALCULUS

In this section, we briefly review the geometrical formulation of constrained variational calculus along the lines described in [8]. The basic environment is a  $(n+1)$ -dimensional fiber bundle  $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$ , referred to local fibred coordinates  $t, q^1, \dots, q^n$  and called the *event space*. Every section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ , locally described as  $q^i = q^i(t)$ , is interpreted as an evolution of an abstract system  $\mathcal{B}$  with  $n$  degrees of freedom: for instance, if  $\mathcal{B}$  represents a mechanical system, the manifold  $\mathcal{V}_{n+1}$  is identified with the associated *configuration space-time*, and the fibration  $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$  with the absolute time function.

The first jet bundle  $j_1(\mathcal{V}_{n+1})$ , referred to local jet coordinates  $t, q^i, \dot{q}^i$ , is called the *velocity space*. Every section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  admits a corresponding lift  $j_1(\gamma) : \mathbb{R} \rightarrow j_1(\mathcal{V}_{n+1})$ , locally expressed as  $q^i = q^i(t), \dot{q}^i = \frac{dq^i}{dt}$ .

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The presence of non-holonomic constraints is geometrized through the assignment of a submanifold  $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$  fibred over  $\mathcal{V}_{n+1}$ , as described by the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{V}_{n+1} & \xlongequal{\quad} & \mathcal{V}_{n+1} \end{array} \quad (\text{II.1})$$

all vertical arrows denoting bundle projections. Referring  $\mathcal{A}$  to local fibered coordinates  $t, q^i, z^A$  ( $A = 1, \dots, r < n$ ), the embedding  $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$  is locally represented as

$$\dot{q}^i = \psi^i(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad (\text{II.2})$$

with  $\text{rank} \left\| \frac{\partial \psi^i}{\partial z^A} \right\| = r$ .

A section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  is called *admissible* if and only if there exists a section  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$  satisfying  $j_1(\gamma) = i \cdot \hat{\gamma}$ . A section  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$  is similarly called admissible if and only if  $i \cdot \hat{\gamma} = j_1(\pi \cdot \hat{\gamma})$ . In coordinates, if  $\hat{\gamma}$  is described as  $q^i = q^i(t), z^A = z^A(t)$ , the admissibility condition is summarized into the system of first order ODE's

$$\frac{dq^i}{dt} = \psi^i(t, q^1(t), \dots, q^n(t), z^1(t), \dots, z^r(t)) \quad (\text{II.3})$$

The geometry of the submanifold  $\mathcal{A}$  has been extensively studied in the context of non-holonomic mechanics (see, among others, [9, 10] and references therein). For the present purposes, we recall the concept of *contact bundle*  $\pi : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$ , meant as the vector sub-bundle of the cotangent space  $T^*(\mathcal{A})$  locally spanned by the contact 1-forms

$$\omega^i := dq^i - \psi^i(t, q^k, z^A) dt \quad (\text{II.4})$$

Denoting by  $V(\mathcal{V}_{n+1}) \subset T(\mathcal{V}_{n+1})$  the vertical bundle relative to the fibration  $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$  and by  $V^*(\mathcal{V}_{n+1})$  the associated dual bundle — commonly referred to as the *phase space* — the manifold  $\mathcal{C}(\mathcal{A})$  is canonically isomorphic to the pull-back of  $V^*(\mathcal{V}_{n+1})$  through the fibered morphism

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\pi} & V^*(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \quad (\text{II.5})$$

We refer  $\mathcal{C}(\mathcal{A})$  to fibered coordinates  $t, q^i, z^A, p_i$ , defined according to the identification  $\sigma = p_i(\sigma) \omega^i|_{\pi(\sigma)} \forall \sigma \in \mathcal{C}(\mathcal{A})$ . An important geometrical attribute of the contact bundle is the its *Liouville 1-form*  $\Theta$ , locally expressed as [16]

$$\Theta := p_i \omega^i = p_i (dq^i - \psi^i(t, q^k, z^A) dt) \quad (\text{II.6})$$

The geometrical framework outlined above provides the mathematical setting for an intrinsic formulation of constrained variational calculus. To this end, we consider an action functional of the form

$$\mathcal{I}[\gamma] := \int_{\hat{\gamma}} L dt = \int_{t_0}^{t_1} L(t, q^i(t), z^A(t)) dt \quad (\text{II.7})$$

assigning to each admissible section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  a corresponding “cost”, expressed as the integral of a *Lagrangian function*  $L(t, q^i, z^A) \in \mathcal{F}(\mathcal{A})$  along the lift  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ . The aim is studying the (local) extremals of the functional (II.7) with respect to admissible deformations of  $\gamma$  leaving the endpoints  $\gamma(t_0), \gamma(t_1)$  fixed.

This may be achieved observing that, under very general assumptions, the original problem is mathematically equivalent to a *free* variational problem on the contact bundle  $\pi : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$ . The procedure, outlined in [8], relies on the fact that, by means of the Liouville 1-form (II.6), every Lagrangian  $L(t, q^i, z^A) \in \mathcal{F}(\mathcal{A})$  may be lifted to a 1-form  $\vartheta_L$  over  $\mathcal{C}(\mathcal{A})$  according to the prescription

$$\vartheta_L := L dt + \Theta = (L - p_i \psi^i) dt + p_i dq^i := -\mathcal{H} dt + p_i dq^i \quad (\text{II.8})$$

The function  $\mathcal{H}(t, q^k, z^A, p_k) = -L(t, q^k, z^A) + p_i \psi^i(t, q^k, z^A) \in \mathcal{F}(\mathcal{C}(\mathcal{A}))$  is known in the literature as the *Pontryagin Hamiltonian*. By means of the 1-form (II.8), to each section  $\bar{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$ , expressed in coordinates as

$q^i = q^i(t)$ ,  $z^A = z^A(t)$ ,  $p_i = p_i(t)$ ), we assign the action functional

$$\bar{\mathcal{I}}[\bar{\gamma}] := \int_{\bar{\gamma}} \vartheta_L = \int_{t_0}^{t_1} \left[ L(t, q^k(t), z^A(t)) + p_i(t) \left( \frac{dq^i}{dt} - \psi^i(t, q^k(t), z^A(t)) \right) \right] dt \quad (\text{II.9})$$

The resulting setup is closely related to the original problem based on the functional (II.7) and on the constraints (II.3). In fact, denoting by  $\nu : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{V}_{n+1}$  the composite projection  $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{V}_{n+1}$ , it turns out that every “ordinary” extremal of the original problem is the projection  $\gamma = \nu \cdot \bar{\gamma}$  of a solutions of the free variational problem based on the functional (II.9) [17]. More specifically, the requirement of stationarity of the action integral (II.9) under arbitrary deformations leaving the projections  $\nu(\bar{\gamma}(t_0))$ ,  $\nu(\bar{\gamma}(t_1))$  fixed leads to  $2n + r$  equations

$$\frac{dq^i}{dt} = \psi^i(t, q^k, z^A) = \frac{\partial \mathcal{H}}{\partial p_i} \quad (\text{II.10a})$$

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i} - p_k \frac{\partial \psi^k}{\partial q^i} = - \frac{\partial \mathcal{H}}{\partial q^i} \quad (\text{II.10b})$$

$$p_i \frac{\partial \psi^i}{\partial z^A} - \frac{\partial L}{\partial z^A} = \frac{\partial \mathcal{H}}{\partial z^A} = 0 \quad (\text{II.10c})$$

for the unknowns  $q^i(t)$ ,  $z^A(t)$ ,  $p_i(t)$ , identical to the *Pontryagin equations* [11, 12] involved in the study of the constrained functional (II.7).

As far as the ordinary extremals are concerned, the original constrained variational problem in the event space is therefore equivalent to a free variational problem in the contact bundle.

In order to analyse the content of the system (II.10), it is convenient to start with eq. (II.10c). The latter identifies a subset of  $\mathcal{C}(\mathcal{A})$ , henceforth denoted by  $\mathcal{S}$ . The Hamiltonian  $\mathcal{H}$  is called *regular* if and only if the condition

$$\det \left( \frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B} \right)_\sigma \neq 0 \quad (\text{II.11})$$

holds for all  $\sigma \in \mathcal{S}$ . When this is the case, eqs. (II.10c) may be uniquely solved for the variables  $z^A$ , giving rise to a representation of the form

$$z^A = z^A(t, q^i, p_i) \quad (\text{II.12})$$

Under the stated assumption, the subset  $\mathcal{S}$  is therefore a  $(2n + 1)$ -dimensional submanifold  $i : \mathcal{S} \rightarrow \mathcal{C}(\mathcal{A})$ , locally diffeomorphic to the phase space  $V^*(\mathcal{V}_{n+1})$ .

The pull-back  $H := i^*(\mathcal{H})$  of the Pontryagin Hamiltonian  $\mathcal{H}$ , expressed in coordinates as

$$H(t, q^i, p_i) := \mathcal{H}(t, q^i, z^A(t, q^k, p_k), p_i) = p_h \psi^h(t, q^i, z^A(t, q^k, p_k)) - L(t, q^i, z^A(t, q^k, p_k)) \quad (\text{II.13})$$

yields a proper Hamiltonian function on  $\mathcal{S}$ . Through the latter, the remaining equations (II.10) may be written as ordinary Hamilton equations. On account of eqs. (II.10c) we have in fact the identifications

$$\frac{\partial H}{\partial p_i} = \frac{\partial \mathcal{H}}{\partial p_i} = \psi^i \quad (\text{II.14a})$$

$$\frac{\partial H}{\partial q^i} = \frac{\partial \mathcal{H}}{\partial q^i} = p_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} \quad (\text{II.14b})$$

allowing to cast eqs. (II.10a), (II.10b) into the form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad (\text{II.15a})$$

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial q^i} \quad (\text{II.15b})$$

The original constrained variational problem is thus reduced to a free Hamiltonian problem in the submanifold  $\mathcal{S}$ , with Hamiltonian  $H(t, q^i, p_i)$  identical to the pull-back  $H = i^*(\mathcal{H})$ .

### III. THE OSTROGRADSKY PROCEDURE REVISITED

In this section, we propose a revisitation of Ostrogradsky's construction of a Hamiltonian setup for the study of variational problems based on non-degenerate Lagrangians  $L(t, q^i, \dot{q}^i, \ddot{q}^i, \dots)$  involving higher order derivatives of the configuration variables [1, 7]. The idea is regarding any such  $L$  as a function on a submanifold of a suitable velocity space, thereby reducing the original problem to a constrained one, of the kind described in the Section II.

As we shall see, pursuing this viewpoint will provide an identification of the Ostrogradsky Hamiltonian with the pull-back (II.13) of the Pontryagin one, thus opening the way to a self-consistent interpretation of Ostrogradsky's formalism in modern geometrical terms.

For the sake of simplicity, and to better fix the basic ideas and notations, we shall first consider Lagrangians of order 2 in the derivatives. The procedure will then be extended to higher order Lagrangians.

#### A. Lagrangians of order 2 in the derivatives

To start with, let us briefly review the Ostrogradsky procedure for Lagrangians of derivative order 2. Given a Lagrangian of the form  $L(t, q^i, \dot{q}^i, \ddot{q}^i)$ , the associated Euler–Lagrange equations read

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}^i} = 0, \quad i = 1, \dots, n \quad (\text{III.16})$$

Assuming the validity of non-degeneracy condition  $\det \left\| \frac{\partial^2 L}{\partial \ddot{q}^i \partial \ddot{q}^j} \right\| \neq 0$ , Ostrogradsky's idea consists in adopting the functions

$$q^i, \quad \dot{q}^i, \quad p_i^0 := \frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}^i}, \quad p_i^1 := \frac{\partial L}{\partial \ddot{q}^i} \quad (\text{III.17})$$

as coordinates in a  $(4n + 1)$ -dimensional *phase space*, with  $p_i^0$  and  $p_i^1$  respectively meant as canonical momenta conjugate to the variables  $\dot{q}^i$  and  $\ddot{q}^i$ .

Under the stated non-degeneracy condition, the last set of equations (III.17) can be solved for the unknowns  $\ddot{q}^i$ , giving rise to a representation of the form

$$\ddot{q}^i = \ddot{q}^i(t, q^k, \dot{q}^k, p_k^1) \quad (\text{III.18})$$

In this way, introducing the *Ostrogradsky Hamiltonian*

$$H := p_i^0 \dot{q}^i + p_i^1 \ddot{q}^i - L \quad (\text{III.19})$$

and expressing it in terms of the variables  $q^i, \dot{q}^i, p_i^0, p_i^1$  through eqs. (III.18), a straightforward calculation yields the relations

$$\frac{\partial H}{\partial p_i^0} = \dot{q}^i, \quad \frac{\partial H}{\partial p_i^1} = \ddot{q}^i, \quad \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i}, \quad \frac{\partial H}{\partial \dot{q}^i} = p_i^0 - \frac{\partial L}{\partial \dot{q}^i}$$

In view of these, the content of eqs. (III.16), (III.17) may be cast into the Hamiltonian form

$$\frac{dq^i}{dt} = \dot{q}^i = \frac{\partial H}{\partial p_i^0}, \quad \frac{d\dot{q}^i}{dt} = \ddot{q}^i = \frac{\partial H}{\partial p_i^1} \quad (\text{III.20a})$$

$$\frac{dp_i^0}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}^i} = \frac{\partial L}{\partial q^i} = -\frac{\partial H}{\partial q^i}, \quad \frac{dp_i^1}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}^i} = \frac{\partial L}{\partial \dot{q}^i} - p_i^0 = -\frac{\partial H}{\partial \dot{q}^i} \quad (\text{III.20b})$$

To clarify the geometrical meaning of the Ostrogradsky procedure, we focus on the fiber bundle  $\mathcal{V}_{n+1} \rightarrow \mathbb{R}$  and on the associated first jet bundle. For reasons that will be clear soon, we change the notation  $j_1(\mathcal{V}_{n+1})$  into  $\mathcal{Q}$ ,  $q^i$  into  $q_0^i$ ,  $\dot{q}^i$  into  $q_1^i$ , and regard the bundle  $t: \mathcal{Q} \rightarrow \mathbb{R}$ , referred to local coordinates  $t, q_\alpha^i$ ,  $\alpha = 0, 1$  as our new event space.

By its very definition, the second jet bundle  $j_2(\mathcal{V}_{n+1})$  is then (canonically isomorphic to) an affine subbundle of the first jet bundle  $j_1(\mathcal{Q})$ , as expressed by the commutative diagram

$$\begin{array}{ccc} j_2(\mathcal{V}_{n+1}) & \xrightarrow{i} & j_1(\mathcal{Q}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{Q} & \xlongequal{\quad} & \mathcal{Q} \end{array} \quad (\text{III.21})$$

Referring  $j_1(\mathcal{Q})$  to jet coordinates  $t, q_\alpha^i, \dot{q}_\alpha^i$ , the image  $i(j_2(\mathcal{V}_{n+1})) \subset j_1(\mathcal{Q})$ , henceforth denoted by  $\mathcal{A}$ , is locally described by the equations  $\dot{q}_0^i = q_1^i$ . We can therefore refer  $\mathcal{A}$  to local fibred coordinates  $t, q_\alpha^i, z^i$ , and represent the imbedding  $\mathcal{A} \rightarrow j_1(\mathcal{Q})$  through the equations (analogous to eqs. (II.2))

$$\dot{q}_\alpha^i = \psi_\alpha^i(t, q_0^i, q_1^i, z^i), \quad (\text{III.22})$$

with  $\psi_0^i = q_1^i$  and  $\psi_1^i = z^i$ .

Alternatively, we may regard  $\mathcal{A}$  as a fiber bundle over  $\mathcal{V}_{n+1}$ , related to  $j_2(\mathcal{V}_{n+1})$  by the fibred isomorphism  $(t, q_0^i, q_1^i, z^i) \longleftrightarrow (t, q^i, \dot{q}^i, \ddot{q}^i)$ .

Collecting all results, we conclude that assigning a variational problem in  $\mathcal{V}_{n+1}$ , based on a Lagrangian  $L(t, q^i, \dot{q}^i, \ddot{q}^i) \in \mathcal{F}(j_2(\mathcal{V}_{n+1}))$ , is equivalent to assigning a constrained variational problem in  $\mathcal{Q}$ , with constraint submanifold  $\mathcal{A} \rightarrow j_1(\mathcal{Q})$  described by eqs. (III.22) and Lagrangian  $L(t, q_\alpha^i, z^i) \in \mathcal{F}(\mathcal{A})$ .

In the determination of the extremals, we can therefore proceed along the lines developed in Section II. To this end, we consider once again the contact bundle  $\mathcal{C}(\mathcal{A})$  over  $\mathcal{A}$ , and denote by  $\nu : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{Q}$  the composite projection  $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{Q}$ , and by  $t, q_\alpha^i, z^i, p_i^\alpha$  the local coordinates on  $\mathcal{C}(\mathcal{A})$  defined by the prescription

$$\sigma = p_i^\alpha(\sigma) (dq_\alpha^i - \psi_\alpha^i dt)_{|\pi(\sigma)} \quad \forall \sigma \in \mathcal{C}(\mathcal{A}) \quad (\text{III.23})$$

the summation convention being henceforth extended to all type of indices.

Starting with the Lagrangian  $L(t, q_\alpha^i, z^i) \in \mathcal{F}(\mathcal{A})$ , we then construct the 1-form

$$\vartheta_L = L dt + p_i^\alpha (dq_\alpha^i - \psi_\alpha^i dt) = -\mathcal{H} dt + p_i^\alpha dq_\alpha^i \in \mathcal{C}(\mathcal{A}) \quad (\text{III.24})$$

with

$$\mathcal{H}(t, q_\alpha^i, z^i, p_i^\alpha) = p_i^\alpha \psi_\alpha^i - L(t, q_\alpha^i, z^i) = p_i^0 q_1^i + p_i^1 z^i - L(t, q_\alpha^i, z^i) \quad (\text{III.25})$$

denoting the Pontryagin Hamiltonian. Eventually, we assign to each section  $\bar{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  the action functional

$$\bar{\mathcal{I}}[\bar{\gamma}] := \int_{\bar{\gamma}} \vartheta_L = \int_{t_0}^{t_1} \left( -\mathcal{H}(t, q_\alpha^i, z^i, p_i^\alpha) + p_i^\alpha \frac{dq_\alpha^i}{dt} \right) dt \quad (\text{III.26})$$

The request of stationarity of the integral (III.26) under arbitrary deformations leaving the points  $\nu(\bar{\gamma}(t_0))$ ,  $\nu(\bar{\gamma}(t_1))$  fixed leads to the Pontryagin equations

$$\frac{dq_\alpha^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i^\alpha}, \quad \frac{dp_i^\alpha}{dt} = -\frac{\partial \mathcal{H}}{\partial q_\alpha^i} \quad (\text{III.27a})$$

$$\frac{\partial \mathcal{H}}{\partial z^i} = p_i^1 - \frac{\partial L}{\partial z^i} = 0 \quad (\text{III.27b})$$

for the unknowns  $q_\alpha^i(t), z^i(t), p_i^\alpha(t)$ . These are the precise analogue of eqs. (II.10) for the case in study. In particular, eqs. (III.27b) reproduce the content of the last set of eqs. (III.17) under the morphism  $(t, q_0^i, q_1^i, z^i) \longleftrightarrow (t, q^i, \dot{q}^i, \ddot{q}^i)$ .

Denoting by  $\mathcal{S}$  the subset of  $\mathcal{C}(\mathcal{A})$  described by eqs. (III.27b) and taking eq. (III.25) into account, it is readily seen that the non-degeneracy condition  $\det \left\| \frac{\partial^2 L}{\partial \ddot{q}^i \partial \ddot{q}^j} \right\| \neq 0$ , here rephrased as  $\det \left\| \frac{\partial^2 L}{\partial z^i \partial z^j} \right\| \neq 0$ , automatically ensures the regularity of the Pontryagin Hamiltonian (III.25). We can therefore solve eqs. (III.27b) for the variables  $z^i$ , getting an expression of the form

$$z^i = z^i(t, q_\alpha^k, p_k^1) \quad (\text{III.28})$$

formally identical to eq. (III.18)

Exactly as it happened in Section II, eq. (III.28) allows to regard  $\mathcal{S}$  as a submanifold  $i : \mathcal{S} \rightarrow \mathcal{C}(\mathcal{A})$ , locally diffeomorphic to the phase space  $V^*(\mathcal{Q})$ . In view of eqs. (III.17), (III.27b), the pull back of the Pontryagin Hamiltonian (III.25) to the submanifold  $\mathcal{S}$  yields the function

$$H(t, q_\alpha^i, p_i^\alpha) := p_i^0 q_1^i + p_i^1 z^i(t, q_\alpha^k, p_k^1) - L(t, q_\alpha^i, z^i(t, q_\alpha^k, p_k^1)) \quad (\text{III.29})$$

identical to the Ostrogradsky Hamiltonian (III.19) and satisfying the relations

$$\frac{\partial H}{\partial p_i^\alpha} = \frac{\partial \mathcal{H}}{\partial p_i^\alpha}, \quad \frac{\partial H}{\partial q_\alpha^i} = \frac{\partial \mathcal{H}}{\partial q_\alpha^i} \quad (\text{III.30})$$

On account of the latter, eqs. (III.27a) may be cast into the canonical Hamiltonian form

$$\frac{dq_0^i}{dt} = \frac{\partial H}{\partial p_i^0}, \quad \frac{dq_1^i}{dt} = \frac{\partial H}{\partial p_i^1} \quad (\text{III.31a})$$

$$\frac{dp_i^0}{dt} = -\frac{\partial H}{\partial q_0^i}, \quad \frac{dp_i^1}{dt} = -\frac{\partial H}{\partial q_1^i} \quad (\text{III.31b})$$

identical to the one taken by the Ostrogradsky equations (III.20).

### B. Lagrangians of order $N$ in the derivatives

The Ostrogradsky construction is easily extended to Lagrangians depending on higher order derivatives. To this end, let  $j_N(\mathcal{V}_{n+1})$  denote the  $N^{\text{th}}$  jet-bundle of the event space, referred to fibred coordinates  $t, q^i, q_1^i, \dots, q_N^i$ . Setting  $q_0^i = q^i$ , the Euler–Lagrange equations associated with a Lagrangian  $L(t, q^i, q_1^i, \dots, q_N^i)$  are synthetically written as

$$\sum_{\alpha=0}^N (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \frac{\partial L}{\partial q_\alpha^i} = 0, \quad i = 1, \dots, n \quad (\text{III.32})$$

For each  $\alpha = 0, \dots, N-1$ , let the canonical momentum  $p_i^\alpha$  conjugate to the coordinate  $q_\alpha^i$  be defined according to the prescription

$$p_i^\alpha := \sum_{\beta=\alpha}^{N-1} (-1)^{\beta-\alpha} \frac{d^{\beta-\alpha}}{dt^{\beta-\alpha}} \frac{\partial L}{\partial q_{\beta+1}^i} \quad (\text{III.33})$$

whence, in particular

$$p_i^{N-1} := \frac{\partial L}{\partial q_N^i}(t, q^i, q_1^i, \dots, q_N^i) \quad (\text{III.33}')$$

The variables  $t, q_\alpha^i, p_i^\alpha$ ,  $\alpha = 0, \dots, N-1$  are regarded as coordinates in a  $(2nN+1)$ -dimensional *phase space*. Under the non-degeneracy assumption  $\det \left\| \frac{\partial^2 L}{\partial q_N^i \partial q_N^j} \right\| \neq 0$ , eqs. (III.33') may be solved for  $q_N^i$ , giving rise to expressions of the form

$$q_N^i = q_N^i(t, q_0^k, \dots, q_{N-1}^k, p_k^{N-1}) \quad (\text{III.34})$$

The Ostrogradsky Hamiltonian is then defined as

$$H(t, q_0^i, \dots, q_{N-1}^i, p_i^0, \dots, p_i^{N-1}) := \sum_{\alpha=0}^{N-1} p_i^\alpha q_{\alpha+1}^i - L(t, q_0^i, \dots, q_{N-1}^i, q_N^i) \quad (\text{III.35})$$

with  $q_N^i$  given by eq. (III.34). In this way, taking eqs. (III.33') into account, the Hamilton equations generated by the Hamiltonian (III.35) are easily recognized to be

$$\frac{dq_\alpha^i}{dt} = \frac{\partial H}{\partial p_i^\alpha} = q_{\alpha+1}^i \quad (\alpha = 0, \dots, N-2), \quad \frac{dq_{N-1}^i}{dt} = \frac{\partial H}{\partial p_i^{N-1}} = q_N^i(t, q_0^k, \dots, q_{N-1}^k, p_k^{N-1}) \quad (\text{III.36a})$$

$$\frac{dp_i^0}{dt} = -\frac{\partial H}{\partial q_0^i} = -\frac{\partial L}{\partial q_0^i}, \quad \frac{dp_i^\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha^i} = -p_i^{\alpha-1} + \frac{\partial L}{\partial q_\alpha^i} \quad (\alpha = 2, \dots, N) \quad (\text{III.36b})$$

Conversely, a straightforward check shows that eqs. (III.36b), together with eqs. (III.33'), imply the validity of the Euler–Lagrange equations (III.32).

A deeper insight into the geometrical meaning of the Ostrogradsky algorithm is gained denoting by  $Q := j_{N-1}(\mathcal{V}_{n+1})$  the  $(N-1)^{\text{th}}$  jet bundle of the fibration  $t : \mathcal{V}_{n+1} \rightarrow \mathbb{R}$ , regarded as a fibre bundle  $t : \mathcal{Q} \rightarrow \mathbb{R}$ , and by  $j_1(\mathcal{Q})$  the

corresponding first jet bundle. The  $N^{th}$  jet bundle  $j_N(\mathcal{V}_{n+1})$  is then canonically isomorphic to an affine subbundle of  $j_1(\mathcal{Q})$ , as summarized into the commutative diagram

$$\begin{array}{ccc} j_N(\mathcal{V}_{n+1}) & \xrightarrow{i} & j_1(\mathcal{Q}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{Q} & \xlongequal{\quad} & \mathcal{Q} \end{array}$$

Adopting  $t, q_\alpha^i$ , ( $\alpha = 0, \dots, N-1$ ) as local coordinates in  $\mathcal{Q}$ , and referring  $j_1(\mathcal{Q})$  to jet coordinates  $t, q_\alpha^i, \dot{q}_\alpha^i$ , the submanifold  $\mathcal{A} := i(j_N(\mathcal{V}_{n+1})) \subset j_1(\mathcal{Q})$  is locally described by the equations  $\dot{q}_\alpha^i = q_{\alpha+1}^i$ ,  $\alpha = 0, \dots, N-2$ .

We can therefore refer  $\mathcal{A}$  to local fibred coordinates  $t, q_0^i, \dots, q_{N-1}^i, z^i$ , and represent the imbedding  $\mathcal{A} \rightarrow j_1(\mathcal{Q})$  through the equations

$$\dot{q}_\alpha^i = \psi_\alpha^i(t, q_0^i, \dots, q_{N-1}^i, z^i), \quad \alpha = 0, \dots, N-1 \quad (\text{III.37})$$

with  $\psi_\alpha^i = q_{\alpha+1}^i$ ,  $\alpha = 0, \dots, N-2$ , and  $\psi_{N-1}^i = z^i$ . Alternatively, we may regard  $\mathcal{A}$  as a fiber bundle over  $\mathcal{V}_{n+1}$ , isomorphic to  $j_N(\mathcal{V}_{n+1})$  through the fibred morphism  $(t, q_0^i, \dots, q_{N-1}^i, z^i) \longleftrightarrow (t, q_0^i, \dots, q_{N-1}^i, q_N^i)$ .

Once again, collecting all results, we conclude that assigning a variational problem in  $\mathcal{V}_{n+1}$ , based on a Lagrangian  $L(t, q_0^i, \dots, q_N^i) \in \mathcal{F}(j_N(\mathcal{V}_{n+1}))$ , is equivalent to assigning a constrained variational problem in  $\mathcal{Q}$ , with constraint submanifold  $\mathcal{A} \rightarrow j_1(\mathcal{Q})$  described by eqs. (III.37) and Lagrangian  $L(t, q_0^i, \dots, q_{N-1}^i, z^i) \in \mathcal{F}(\mathcal{A})$ .

The constrained problem in  $\mathcal{Q}$  may then be lifted to a free variational problem on the contact bundle  $\mathcal{C}(\mathcal{A})$ , referred to fibred coordinates  $t, q_\alpha^i, z^i, p_i^\alpha$ ,  $\alpha = 0, \dots, N-1$ . The procedure, identical to the one exploited in Sect. III A, culminates in the introduction of the Pontryagin Hamiltonian

$$\mathcal{H}(t, q_\alpha^i, z^i, p_i^\alpha) = \sum_{\alpha=0}^{N-1} p_i^\alpha \psi_\alpha^i - L(t, q_\alpha^i, z^i) = \sum_{\alpha=0}^{N-2} p_i^\alpha q_{\alpha+1}^i + p_i^{N-1} z^i - L(t, q_\alpha^i, \dots, q_{N-1}^i, z^i) \quad (\text{III.38})$$

Preserving the notation  $\mathcal{C}(\mathcal{A}) \xrightarrow{\nu} \mathcal{Q}$  for the composite map  $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{Q}$ , to each section  $\bar{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  we now assign the action functional

$$\bar{\mathcal{I}}[\bar{\gamma}] := \int_{\bar{\gamma}} -\mathcal{H} dt + \sum_{\alpha=0}^{N-1} p_i^\alpha dq_\alpha^i = \int_{t_0}^{t_1} \left( -\mathcal{H} + \sum_{\alpha=0}^{N-1} p_i^\alpha \frac{dq_\alpha^i}{dt} \right) dt \quad (\text{III.39})$$

Imposing stationarity of the latter under arbitrary deformations leaving the projections  $\nu(\bar{\gamma}(t_0))$ ,  $\nu(\bar{\gamma}(t_1))$  fixed leads to the Pontryagin equations

$$\frac{dq_\alpha^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i^\alpha}, \quad \frac{dp_i^\alpha}{dt} = -\frac{\partial \mathcal{H}}{\partial q_\alpha^i} \quad (\text{III.40a})$$

$$\frac{\partial \mathcal{H}}{\partial z^i} = p_i^{N-1} - \frac{\partial L}{\partial z^i} = 0 \quad (\text{III.40b})$$

for the unknowns  $q_\alpha^i(t), z^i(t), p_i^\alpha(t)$ . Eqs. (III.40b) reproduce the content of eqs. (III.33') under the morphism  $(t, q_0^i, \dots, q_{N-1}^i, z^i) \longleftrightarrow (t, q_0^i, \dots, q_{N-1}^i, q_N^i)$ .

Denoting by  $\mathcal{S}$  the subset of  $\mathcal{C}(\mathcal{A})$  described by eqs. (III.40b), it is readily seen that the non-degeneracy condition  $\left\| \frac{\partial^2 L}{\partial q_N^i \partial q_N^j} \right\| \neq 0$ , here rephrased as  $\det \left\| \frac{\partial^2 L}{\partial z^i \partial z^j} \right\| \neq 0$ , automatically ensures the regularity of the Pontryagin Hamiltonian (III.38). We can therefore solve eqs. (III.40b) for the variables  $z^i$ , getting the expression

$$z^i = z^i(t, q_0^k, \dots, q_{N-1}^k, p_k^{N-1}) \quad (\text{III.41})$$

formally identical to eq. (III.34).

Eqs. (III.41) point out that the subset  $\mathcal{S} \subset \mathcal{C}(\mathcal{A})$  is in fact a *submanifold*  $i : \mathcal{S} \rightarrow \mathcal{C}(\mathcal{A})$ , locally diffeomorphic to the phase space  $V^*(Q)$ . A straightforward check shows that the pull back  $H := i^*(\mathcal{H})$  of the Pontryagin Hamiltonian (III.38) determines a proper Hamiltonian function on  $V^*(Q)$ , identical to the Ostrogradsky Hamiltonian (III.35), and that the Hamilton equations generated by  $H$  coincide with the Ostrogradsky equations (III.36).

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